$$\frac{d}{dx}\left[f(x)^{g(x)}\right]$$

x^n $\quad$ b^x

$\underline{(f(x))^n}$ $\quad$ $\underline{b^{f(x)}}$

$$y = f(x)^{g(x)}$$ use logarithmic differentiation

$$\hookrightarrow \ln y = \ln\left(f(x)^{g(x)}\right) = g(x) \cdot \ln(f(x))$$

implicit diff.

$$\hookrightarrow \frac{1}{y} \cdot y' = g'(x)\ln(f(x)) + g(x) \cdot \frac{f'(x)}{f(x)}$$

$$\hookrightarrow y' = y\left(g'(x)\ln(f(x)) + g(x)\frac{f'(x)}{f(x)}\right)$$

$$= f(x)^{g(x)}\left(g'(x)\ln(f(x)) + g(x)\frac{f'(x)}{f(x)}\right)$$

$$\frac{d}{dx}\left[f(x)^{g(x)}\right] = f(x)^{g(x)}\left(\underbrace{g'(x)\ln(f(x))}_{\text{"exponential derivative part"}} + \underbrace{g(x)\frac{f'(x)}{f(x)}}_{\text{"power rule part"}}\right)$$

Ex: Find $f'(x)$.

① $f(x) = x^x$, $f'(x) = x^x(\ln x + 1)$

$$y = x^x \longrightarrow \ln y = \ln(x^x) = x\ln x$$

$$\frac{d}{dx}(\ln y = x\ln x)$$

$$\hookrightarrow \frac{1}{y}\cdot y' = 1\cdot \ln x + x\cdot \frac{1}{x} = \ln x + 1$$

$$y' = y(\ln x + 1)$$

$$= \boxed{x^x(\ln x + 1) = f'(x)}$$

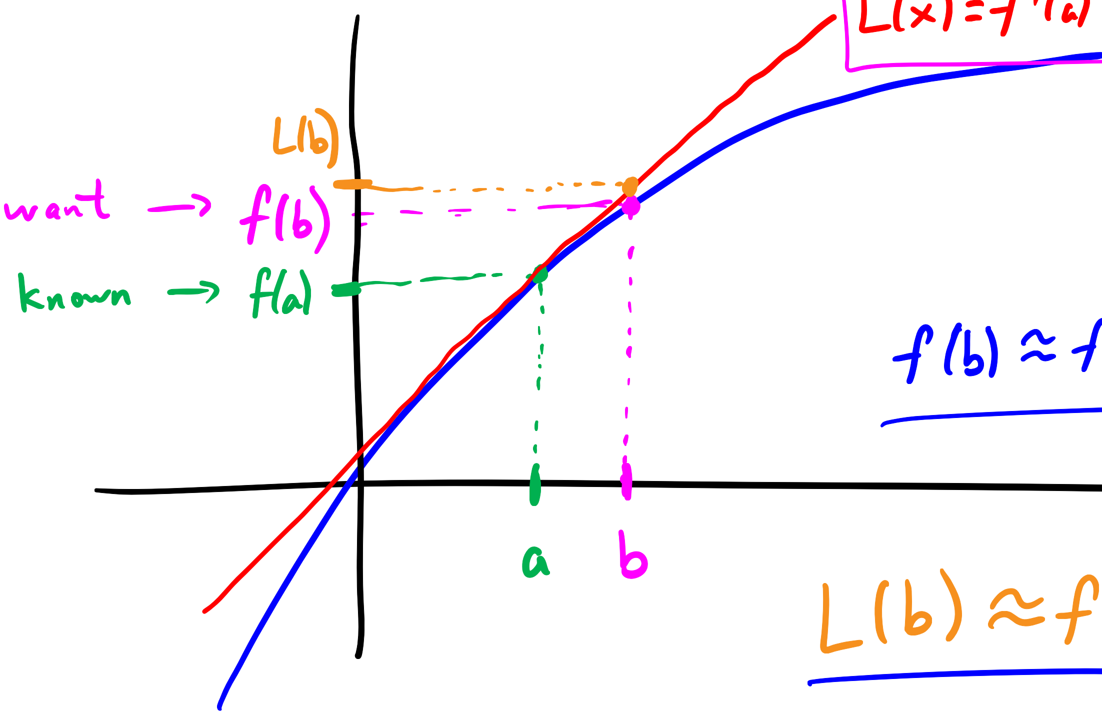
② $f(x) = (\cos x)^{\sin x}$

$$f'(x) = (\cos x)^{\sin x}\left(\cos x \ln(\cos x) - \sin x \tan x\right)$$

Linear Approximation

Approximate a function by its tangent line.

$$\boxed{L(x) = f'(a)(x-a) + f(a)}$$ linearization at $x = a$



$$f(b) \approx f'(a)(b-a) + f(a)$$

$$L(b) \approx f(b)$$

Ex: ① Approximate $\sqrt{103}$

$$f(x) = \sqrt{x}, \quad f(\underbrace{103}_{b}) = \sqrt{103} \approx ?$$

Let $a = 100$, then $f(100) = \sqrt{100} = 10$

$$f'(x) = \frac{1}{2\sqrt{x}}, \quad f'(100) = \frac{1}{20} = 0.05$$

$$f(103) \approx f'(100)(103-100) + f(100)$$

$$= 0.05(3) + 10 = \underline{10.15}$$

$$\underline{(10.148891565)}$$

② Find the linearization of $f(\theta) = \sin\theta$ at $\theta = 0$

$$L(\theta) = f'(0)(\theta - 0) + f(0)$$

$f'(\theta) = \cos\theta$ $\quad f(0) = 0$ $\quad f'(0) = 1$

$$L(\theta) = 1 \cdot (\theta - 0) + 0 = \theta$$

$f(\theta) \approx L(\theta)$ near $\theta = 0$

$\sin\theta \approx \theta$ near $\theta = 0$

Quadratic Approximation (at $x=a$)

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$